二. 有限差分近似基础

1. 网格及有限差分记号

[**双变量函数的有限差分记号 (举例)**] 对于只涉及空间和时间的函数 $u(x,t)$, 考虑初边值问题

$$\begin{cases} \frac{\partial u}{\partial t}=v\frac{\partial^2 u}{\partial x^2}, x\in(0,1), t>0 \\ u(x,0)=f(x), x\in[0,1] \\ u(0,t)=a(t), u(1,t)=b(t), t\geq 0 \\ f(0)=a(0), f(1)=b(0), v>0(v\in C) \end{cases}$$ 的求解.

1. 首先将求解区域用网格划分. 用时间间隔 Δt 和空间间隔 Δx 的两组平行于坐标轴的直线把求解区域网格化, 两组直线的交点称为网格的结点. 用 $u_j^n(j=0,1,...,M)$ 表示 u 在点 $(j\Delta x, n\Delta t)$(记为 (j,n)) 处的近似值;

2. 对问题进行近似, 利用 $\frac{\partial u(x,t)}{\partial t}=\lim\limits_{\Delta t\to 0}\frac{u(x,t+\Delta t)-u(u,t)}{\Delta t}$:

对于 $\frac{\partial u(j\Delta x,n\Delta t)}{\partial t}$ 的合理近似是 $\frac{\partial u(j\Delta x,n\Delta t)}{\partial t}\approx\frac{u_j^{n+1}-u_j^n}{\Delta t}$;

对于 $\frac{\partial^2 u(j\Delta x,n\Delta t)}{\partial x^2}$ 的合理近似是 $\frac{\partial^2 u(j\Delta x,n\Delta t)}{\partial x^2}\approx\frac{\left(\frac{\partial u}{\partial x}\right)_{j+\frac{1}{2}}^n-\left(\frac{\partial u}{\partial x}\right)_{j-\frac{1}{2}}^n}{\Delta x}\approx\frac{\frac{u_{j+1}^n-u_j^n}{\Delta x}-\frac{u_j^n-u_{j-1}^n}{\Delta x}}{\Delta x}\approx\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\Delta x^2}$;

方程的近似为 $$\begin{cases} \frac{u_j^{n+1}-u_j^n}{\Delta t}=v\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\Delta x^2}\Rightarrow u_j^{n+1}=u_j^n+v\frac{\Delta t}{\Delta x^2}(u_{j+1}-2u_j^n+u_{j-1}^n) \\ u_j^0=f(j\Delta x), j=0,1,...,M, \\ u_0^{n+1}=a((n+1)\Delta t), n=0,1,..., \\ u_M^{n+1}=b((n+1)\Delta t), n=0,1,... \end{cases}$$

2. 空间导数近似

[**空间导数差分近似 (基本举例)**] 假定 $u(x,t)$ 对 x 可微, 考虑 $u(x,t)$ 在 t 处对 x 的偏微分近似, t 不变而省略. 记 $x=j\Delta x, u(x+k\Delta x)=u(j\Delta x+k\Delta x)=u_{j+k}$.

对 x 做 Taylor 展开 $u_{j+k}=\sum\limits_{n=0}\left(\frac{\partial^n u}{\partial x^n}\right)_j\frac{1}{n!}(k\Delta x)^n$, 同理 $\begin{cases} u_{j+1}=\sum\limits_{n=0}\left(\frac{\partial^n u}{\partial x^n}\right)_j\frac{1}{n!}\Delta x^n \\ u_{j-1}=\sum\limits_{n=0}\left(\frac{\partial^n u}{\partial x^n}\right)_j\frac{1}{n!}(-\Delta x)^n \end{cases}$.

当 Δx 足够小时, 可取 $\left(\frac{\partial u}{\partial x}\right)_j$ 的合理近似 $\left(\frac{\partial u}{\partial x}\right)_j\approx\begin{cases} \frac{u_{j+1}-u_j}{\Delta x}=\sum\limits_{n=1}\frac{1}{n!}\left(\frac{\partial^n u}{\partial x^n}\right)_j\Delta x^{n-1} \\ \frac{u_j-u_{j-1}}{\Delta x}=\sum\limits_{n=1}\frac{1}{n!}\left(\frac{\partial^n u}{\partial x^n}\right)_j(-\Delta x)^{n-1} \\ \frac{u_{j+1}-u_{j-1}}{2\Delta x}=\sum)n=1\frac{1}{(2n-1)!}\left(\frac{\partial^{2n-1}u}{\partial x^{2n-1}}\right)_j\Delta x^{2n-2} \end{cases}$.

同理 $\left(\frac{\partial^2 u}{\partial x^2}\right)_j\approx\frac{u_{j+1}-2u_j+u_{j-1}}{\Delta x^2}=\sum\limits_{n=1}\frac{2}{(2n)!}\left(\frac{\partial^{2n}u}{\partial x^{2n}}\right)_j\Delta x^{2n-2}$.

[**一阶单边差分近似**] 形如 $\left(\frac{\partial u}{\partial x}\right)_j = \frac{u_{j+1}-u_j}{\Delta x} + O(\Delta x)$ 和 $\left(\frac{\partial u}{\partial x}\right)_j = \frac{u_j-u_{j-1}}{\Delta x} + O(\Delta x)$ 对一阶导数进行近似, 被称为一阶单边差分近似.

[**一阶中心差分近似**] 形如 $\left(\frac{\partial^2 u}{\partial x^2}\right)_j = \frac{u_{j+1}-u_{j-1}}{2\Delta x} + O(\Delta x^2)$ 对一阶导数进行近似, 被称为一阶中心差分近似.

[**更高阶精度的差分近似方法 (待定系数法举例)**] 考虑二阶导数 $\frac{\partial^2 u}{\partial x^2}$ 在点 $x = j\Delta x, x = (j \pm 1)\Delta x, x = (j \pm 2)\Delta x$ 处的 Taylor 展开, 可以得到四阶的近似精度.

$\Delta x^2 \left(\frac{\partial^2 u}{\partial x^2}\right)_j = c_1 u_{j-2} + c_2 u_{j-1} + c_3 u_j + c_4 u_{j+1} + c_5 u_j + O(\Delta x^6)$, 其中 $c_1, ..., c_6$ 待定. 将该式各项在 $x = j\Delta x$ 处 Taylor 展开, 整理得到:

$$
\begin{aligned}
\Delta x^2 \left(\frac{\partial^2 u}{\partial x^2}\right)_j &= (c_1 + c_2 + c_3 + c_4 + c_5) u_j \\
&+ \left[\frac{(-2)}{1!}c_1 + \frac{(-1)}{1!}c_2 + \frac{0}{1!}c_3 + \frac{1}{1!}c_4 + \frac{2}{1!}c_5\right] (\Delta x) \left(\frac{\partial u}{\partial x}\right)_j \\
&+ \left[\frac{(-2)^2}{2!}c_1 + \frac{(-1)^2}{2!}c_2 + \frac{0^2}{2!}c_3 + \frac{1^2}{2!}c_4 + \frac{2^2}{2!}c_5\right] (\Delta x)^2 \left(\frac{\partial^2 u}{\partial x^2}\right)_j \\
&+ \left[\frac{(-2)^3}{3!}c_1 + \frac{(-1)^3}{3!}c_2 + \frac{0^3}{3!}c_3 + \frac{1^3}{3!}c_4 + \frac{2^3}{3!}c_5\right] (\Delta x)^3 \left(\frac{\partial^3 u}{\partial x^3}\right)_j \\
&+ \left[\frac{(-2)^4}{4!}c_1 + \frac{(-1)^4}{4!}c_2 + \frac{0^4}{4!}c_3 + \frac{1^4}{4!}c_4 + \frac{2^4}{4!}c_5\right] (\Delta x)^4 \left(\frac{\partial^4 u}{\partial x^4}\right)_j \\
&+ \left[\frac{(-2)^5}{5!}c_1 + \frac{(-1)^5}{5!}c_2 + \frac{0^5}{5!}c_3 + \frac{1^5}{5!}c_4 + \frac{2^5}{5!}c_5\right] (\Delta x)^5 \left(\frac{\partial^5 u}{\partial x^5}\right)_j \\
&+ O(\Delta x^6) = \sum_{n=1} \frac{2}{(2n)!} \left(\frac{\partial^{2n} u}{\partial x^{2n}}\right)_j \Delta x^{2n}.
\end{aligned}
$$

比较两边系数, 得到线性代数方程组 $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \\ -8 & 1 & 0 & 1 & 8 \\ 16 & 1 & 0 & 1 & 16 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$.

解得 $\begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \end{pmatrix} = \begin{pmatrix} -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} \end{pmatrix}$

因为解使 $\left(\frac{\partial^5 u}{\partial x^5}\right)_j$ 的系数为零, 所以 $\left(\frac{\partial^2 u}{\partial x^2}\right)_j = \frac{-u_{j-2}+16u_{j-2}-30u_j+16u_{j+1}-u_{j+2}}{12\Delta x^2} + O(\Delta x^4)$ 是四阶精度近似.

3. 导数的算子表示

[差分算子]

Δ_x 前差算子: $\Delta_x u_j = u_{j+1} - u_j$;

∇_x 后差算子: $\nabla_x u_j = u_j - u_{j-1}$;

δ_x 中心差分算子: $\delta_x u_j = u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}$;

T_x 移位算子: $T_x u_j = u_{j+1}$;

μ_x 平均算子: $\mu_x u_j = \frac{1}{2}(u_{j+\frac{1}{2}} + u_{j-\frac{1}{2}})$;

D_x 一阶偏导数算子: $D_x = \frac{\partial}{\partial x}$;

I 恒等算子: $I u_j = u_j$.

[差分算子用位移算子的表示]

$\Delta_x = T_x - I, T_x = \Delta_x + I$;

$\nabla_x = I - T_x^{-1}, T_x = (I - \nabla_x)^{-1}$;

$\delta_x = T_x^{\frac{1}{2}} - T_x^{-\frac{1}{2}}$;

$\delta_x^2 = T_x - 2I + T_x^{-1}$;

$\mu_x \delta_x = \frac{1}{2}(T_x - T_x^{-1})$;

$\mu_x^2 = \frac{1}{4}(T_x + 2I + T_x^{-1})$;

$T_x = I + \frac{1}{2}\delta_x^2 + \mu_x \delta_x$. 由 $\delta_x^2 = T_x - 2I + T_x^{-1}$ 和 $\mu_x \delta_x = \frac{1}{2}(T_x - T_x^{-1})$ 得.

[一阶偏导数算子的表示] $D_x = \begin{cases} \frac{1}{h}\ln T_x \\ \frac{1}{h}\sum\limits_{i=1}(-1)^{i+1}\frac{1}{i!}\Delta_x^i \Leftarrow T_x = \Delta_x + I \\ \frac{1}{h}\sum\limits_{i=1}\frac{1}{i!}\nabla_x^i \Leftarrow T_x = (I - \nabla_x)^{-1} \\ \frac{1}{h}\mu_x \sum\limits_{i=1}(-1)^{i+1}\frac{\delta_x^{2i-1}}{(2i-1)!} \Leftarrow \mu_x^2 = \frac{1}{4}\delta_x^2 + I \Leftarrow \delta_x = T_x^{\frac{1}{2}} - T_x^{-\frac{1}{2}} \\ \frac{1}{h}\left[\delta_x - \frac{1}{24}\delta_x^3 + \frac{3}{640}\delta_x^5 ...\right] \Leftarrow \frac{2}{h}\sinh^{-1}\frac{\delta_x}{2} \end{cases}$

对 u_{j+1} 在 u_j 展开: $u_{j+1} = u_j + \frac{h}{1!}\left(\frac{\partial u}{\partial x}\right)_j + \frac{h^2}{2!}\left(\frac{\partial^2 u}{\partial x^2}\right)_j + ... = (I + \frac{h}{1!}D_x + \frac{h^2}{2!}D_x^2 + ...)u_j$.

其中 h 为空间步长. 即 $u_{j+1} = T_x u_j = e^{hD_x}u_j$, 得 $T_x = e^{hD_x}$. 结论 $D_x = \frac{1}{h}\ln T_x$.

[r 阶偏导数算子的表示] $D_x^r = \frac{1}{h^r}\begin{cases} \Delta_x^r - \frac{r}{2}\Delta_x^{r+1} + \frac{r(3r+5)}{24}\Delta_x^{r+2} - ... \\ \nabla_x^r + \frac{r}{2}\nabla_x^{r+1} + \frac{r(3r+5)}{24}\nabla_x^{r+2} + ... \\ \mu_x\delta_x^r - \frac{r+3}{24}\mu_x\delta_x^{r+2} + \frac{5r^2+52r+135}{5760}\mu_x\delta_x^{r+4} - ..., r = 2i+1 \\ \delta_x^2 - \frac{r}{24}\delta_x^{r+2} + \frac{r(5r+22)}{5760}\delta_x^{r+4} - ..., r = 2i \end{cases}$

[**一阶偏导数算子的 Pade 差分近似**] $D_x = \begin{cases} \frac{1}{h}\left[\Delta_x - \frac{\Delta_x^2}{2} + O(\Delta_x^3)\right] = \frac{1}{h}\frac{\Delta_x}{1+\frac{\Delta_x}{2}} + O(h^2) \\ \frac{1}{h}\left[\nabla_x + \frac{\nabla_x^2}{2} + O(\nabla_x^3)\right] = \frac{1}{h}\frac{\nabla_x}{1-\frac{\nabla_x}{2}} + O(h^2) \\ \frac{1}{h}\left[\mu_x\delta_x - \frac{1}{6}\mu_x\delta_x^3 + O(\delta_x^5)\right] = \frac{1}{h}\frac{\mu_x\delta_x}{1+\frac{\delta_x^2}{6}} + O(h^4) \\ \frac{1}{h}\left[\delta_x - \frac{1}{24}\delta_x^3 + O(\delta_x^5)\right] = \frac{1}{h}\frac{\delta_x}{1+\frac{\delta_x^2}{24}} + O(h^4) \end{cases}$.

二阶偏导数算子的 Pade 差分近似: $D_X^2 = \begin{cases} \frac{1}{h^2}\frac{\Delta_x^2}{1+\Delta_x} + O(h^2) \\ \frac{1}{h^2}\frac{\nabla_x^2}{1-\nabla_x} + O(h^2) \\ \frac{1}{h^2}\frac{\mu_x^2\delta_x^2}{1+\frac{1}{3}\delta_x^2} + O(h^4) \\ \frac{1}{h^2}\frac{\delta_x^2}{1+\frac{1}{12}\delta_x^2} + O(h^4) \end{cases}$.

4. 任意阶精度差分格式的建立

[**三点二阶导数中心近似的 Taylor 级数表 (举例)**] 求解 $\left(\frac{\partial^2 u}{\partial x^2}\right)_j - \frac{1}{\Delta x^2}(au_{j-1} + bu_j + cu_{j+1}) =?$

对每一项进行 Taylor 展开得到 Taylor 级数表:

求和项	u_j	$\Delta x\left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3\left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4\left(\frac{\partial^4 u}{\partial x^4}\right)_j$
$\Delta x^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j$	0	0	1	0	0
$-au_{j-1}$	$-a$	$-a(-1)^1\frac{1}{1!}$	$-a(-1)^2\frac{1}{2!}$	$-a(-1)^3\frac{1}{3!}$	$-a(-1)^4\frac{1}{4!}$
$-bu_j$	$-b$	0	0	0	0
$-cu_{j+1}$	$-c$	$-c(1)^1\frac{1}{1!}$	$-c(1)^2\frac{1}{2!}$	$-c(1)^3\frac{1}{3!}$	$-c(1)^4\frac{1}{4!}$

求解 a, b, c, 即 $\begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & -1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} a = 1 \\ b = -2 \\ c = 1 \end{cases}$.

[**截断误差**] 求和不为零的列.

[**截断误差首项**] 第一个不为零的项, 记为 R_j.

对于上面的例子, $R_j = \frac{1}{\Delta x^2}\left(-\frac{a}{24} - \frac{c}{24}\right)\Delta x^4\left(\frac{\partial^4 u}{\partial x^4}\right)_j \Rightarrow R_j = -\frac{\Delta x^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_j$.

5. 非均匀差分网格

[**差分步长**] 定义为 $h_j = x_j - x_{j-1}$.

[**非均匀差分网格 (举例)**] 分析热传导方程 $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ 在步长 $h_j \neq h_{j+1}$ 时的一阶差分格式.

取 $M+1$ 个结点 $a = x_0 < x_1 < ... < x_i < ... < x_M = b$, 将区间 $R = (a,b)$ 分成 M 个单元 $R_j : x_{j-1} \leq x \leq x_j, j = 1, ..., M$.

定义 $x_{j-\frac{1}{2}} = \frac{x_j + x_{j-1}}{2}$, 将 u_j^n 和 $u_{j-\frac{1}{2}}^n$ 在 $x = x_{j-\frac{1}{2}}$ 展开:

$$u_j^n = \sum_{i=0} \frac{(x_j - x_{j-\frac{1}{2}})^i}{i!} \left(\frac{\partial^i u}{\partial x^i}\right)_{j-\frac{1}{2}}^n = \sum_{i=0} \frac{h_j^i}{2^i i!} \left(\frac{\partial^i u}{\partial x^i}\right)_{j-\frac{1}{2}}^n.$$

$$u_{j-1}^n = \sum_{i=0} \frac{(x_{j-1} - x_{j-\frac{1}{2}})^i}{i!} \left(\frac{\partial^i u}{\partial x^i}\right)_{j-\frac{1}{2}}^n = \sum_{i=0} \frac{(-h_j)^i}{2^i i!} \left(\frac{\partial^i u}{\partial x^i}\right)_{j-\frac{1}{2}}^n.$$

两式做差, 得 $u_j^n - u_{j-1}^n = \sum_{i=0} \frac{h_j^i - (-h_j)^i}{2^i i!} \left(\frac{\partial^i u}{\partial x^i}\right)_{j-\frac{1}{2}}^n$.

将差精确到 $O(h^4)$: $u_j^n - u_{j-1}^n = h_j \left(\frac{\partial u}{\partial x}\right)_{j-\frac{1}{2}}^n + \frac{h_j^3}{24} \left(\frac{\partial^3 u}{\partial x^3}\right)_{j-\frac{1}{2}} + O(h^4) \approx h_j \left(\frac{\partial u}{\partial x}\right)_{j-\frac{1}{2}}^n + \frac{h_j^3}{24} \left(\frac{\partial^3 u}{\partial x^3}\right)_j^n + O(h^4)$, 即 $\frac{u_j^n - u_{j-1}^n}{h_j} = \left(\frac{\partial u}{\partial x}\right)_{j-\frac{1}{2}}^n + \frac{h_j^2}{24} \left(\frac{\partial^3 u}{\partial x^3}\right)_j^n + O(h^3)$.

同理, 将 u_{j+1}^n 和 u_j^n 在 $x = x_{j+\frac{1}{2}}$ 展开, 并精确到 $O(h^4)$ 做差.

得 $\frac{u_{j+1}^n - u_j^n}{h_{j+1}} = \left(\frac{\partial u}{\partial x}\right)_{j+\frac{1}{2}}^n + \frac{h_{j+1}^2}{24} \left(\frac{\partial^3 u}{\partial x^3}\right)_j^n + O(h^3)$.

将 $\left(\frac{\partial u}{\partial x}\right)_{j+\frac{1}{2}}^n$ 和 $\left(\frac{\partial u}{\partial x}\right)_{j-\frac{1}{2}}^n$ 在 $x = x_j$ 处展开.

$$\left(\frac{\partial u}{\partial x}\right)_{j+\frac{1}{2}}^n = \sum_{i=0} \frac{h_{j+1}^i}{2^i i!} \left(\frac{\partial^{i+1} u}{\partial x^{i+1}}\right)_j^n.$$

$$\left(\frac{\partial u}{\partial x}\right)_{j-\frac{1}{2}}^n = \sum_{i=0} \frac{(-h_j)^i}{2^i i!} \left(\frac{\partial^{i+1} u}{\partial x^{i+1}}\right)_j^n.$$

做差并精确到 $O(h^3)$ 得: $\left(\frac{\partial u}{\partial x}\right)_{j+\frac{1}{2}}^n - \left(\frac{\partial u}{\partial x}\right)_{j-\frac{1}{2}}^n = \frac{h_{j+1} + h_j}{2} \left(\frac{\partial^2 u}{\partial x^2}\right)_j^n + \frac{h_{j+1}^2 - h_j^2}{8} \left(\frac{\partial^3 u}{\partial x^3}\right)_j^n + O(h^3)$.

求 $\frac{u_{j+1}^n - u_j^n}{h_{j+1}} - \frac{u_j^n - u_{j-1}^n}{h_j} = \left[\left(\frac{\partial u}{\partial x}\right)_{j+\frac{1}{2}}^n - \left(\frac{\partial u}{\partial x}\right)_{j-\frac{1}{2}}^n\right] + \frac{h_{j+1}^2 - h_j^2}{24} \left(\frac{\partial^3 u}{\partial x^3}\right)_j^n + O(h^3)$, 并在两边同时乘 $\frac{2}{h_j + h_{j+1}}$.

得 $\frac{2}{h_j + h_{j+1}} \left(\frac{u_{j+1}^n - u_j^n}{h_{j+1}} - \frac{u_j^n - u_{j-1}^n}{h_j}\right) = \left(\frac{\partial^2 u}{\partial x^2}\right)_j^n + \frac{h_{j+1} - h_j}{3} \left(\frac{\partial^3 u}{\partial x^3}\right)_j^n + O(h^2)$.

方程对 x 的截断误差 $R(x) = \frac{h_{j+1} - h_j}{3} \left(\frac{\partial^3 u}{\partial x^3}\right)_j^n + O(h^2)$.

对时间有 $\frac{u_j^{n+1} - u_j^n}{\Delta t} = \left(\frac{\partial u}{\partial t}\right)_j^n + O(\Delta t)$, 截断误差 $R(t) = O(\Delta t)$.

总截断误差 $R(u) = R(x) + R(t) = \frac{h_{j+1}-h_j}{3}\left(\frac{\partial^3 u}{\partial x^3}\right)_j^n + O(h^2) + O(\Delta t)$.

对于均匀网格 $h_j = h_{j+1}$, 截断误差 $R(u) = O(h^2) + O(\Delta t)$.

原方程的一阶精度差分格式为 $\frac{u_j^{n+1}-u_j^n}{\Delta t} = \frac{2}{h_j+h_{j+1}}\left(\frac{u_{j+1}^n-u_j^n}{h_{j+1}} - \frac{u_j^n-u_{j-1}^n}{h_j}\right)$.

对于均匀网格, 由 $R(u) = O(h^2) + O(\Delta t)$ 得差分格式的精度为 $O(\Delta t + h^2)$ 阶.

6.Fourier 误差分析

[**Fourier 误差分析**] 对于任意周期函数, 都可以写成其 Fourier 分量 e^{ikx} 的形式, 其中 k 为波数. 一个有限差分格式的误差情况可以根据其结果中波数与精确波数的近似情况进行表征.

[**修正波数**] 经过有限差分的 Fourier 分量 ik^*e^{ikx} 中替代精确波数 k 的 k^*.

[**对一阶导数的二阶中心差分格式的 Fourier 误差分析**] 一阶导数 $\left(\frac{\partial u}{\partial x}\right)_j$ 的二阶中心差分格式 $\frac{u_{j+1}-u_{j-1}}{2\Delta x}$ 的修正波数 k^* 精确到精确波数 k 的二阶精度, 有 $k^* = \frac{\sin k\Delta x}{\Delta x}$.

精确的 e^{ikx} 的一阶导数为 $\frac{de^{ikx}}{dx} = ike^{ikx}$.

对 $u_j = e^{ikx_j}$ 应用 $\left(\frac{\partial u}{\partial x}\right)_j = \frac{u_{j+1}-u_{j-1}}{2\Delta x}$ 的二阶中心差分格式, 其中 $x_j = j\Delta x$, 得:

$$\begin{aligned}\left(\frac{\partial u}{\partial x}\right)_j &= \frac{e^{ik\Delta x(j+1)}-e^{ik\Delta x(j-1)}}{2\Delta x} = \frac{e^{ikj\Delta x}}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) \\ &= \frac{e^{ikj\Delta x}}{2\Delta x}\left[(\cos k\Delta x + i\sin k\Delta x) - (\cos k\Delta x - i\sin k\Delta x)\right] = i\frac{\sin k\Delta x}{\Delta x}e^{ikj\Delta x}.\end{aligned}$$

由修正波数的定义 $\left(\frac{\partial u}{\partial x}\right)_j = ik^*e^{ikj\Delta x}$ 得 $k^* = \frac{\sin k\Delta x}{\Delta x}$.

对二阶中心差分格式, k^* 近似 k 到二阶精度. 有 $\frac{\sin k\Delta x}{\Delta x} = k - \frac{k^3\Delta x^2}{3!} + \dots$

TODO

中心差分算子的对称分解